Math 4650

Topic 4 - Limits of functions

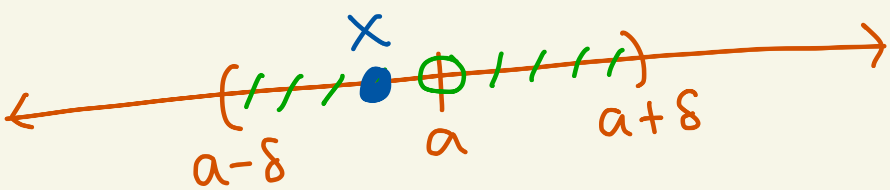
Notation:

We write $f : D \to \mathbb{R}$ to mean that f is a function with domain D and output in $\mathbb{R}$.

Ex: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

Ex: $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$
given by $f(x) = \frac{1}{x}$

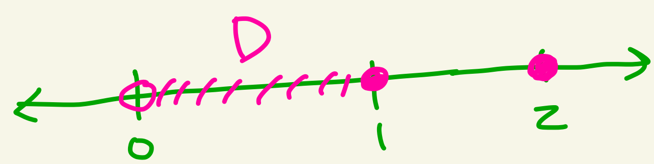
Def: Let $D \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.
We say that a is a <u>limit point of D</u> if for every $\delta > 0$ there exists $x \in D$ with $0 < |x-a| < \delta$.



$0 < |x-a|$ means $x \neq a$

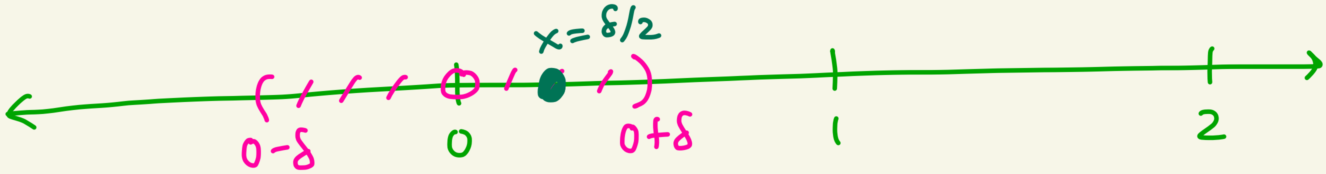
$|x-a| < \delta$ means $a-\delta < x < a+\delta$

Ex: $D = (0,1] \cup \{2\}$



Claim: 0 is a limit point of D.

pf of claim:
Let $\delta > 0$.
Let $x = \min\{\frac{\delta}{2}, \frac{1}{2}\}$.



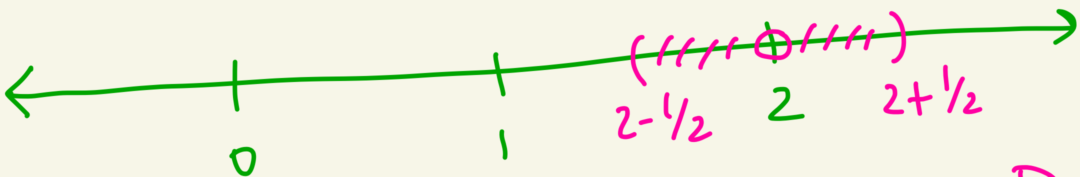
Then, $0 < |x-0| < \delta$ and $x \in D$.
Thus, 0 is a limit point of D.

Claim: 2 is not a limit of D.

Proof of claim:

Let $\delta = \frac{1}{2}$.

There is no $x \in D$ with $0 < |x-2| < \frac{1}{2}$.

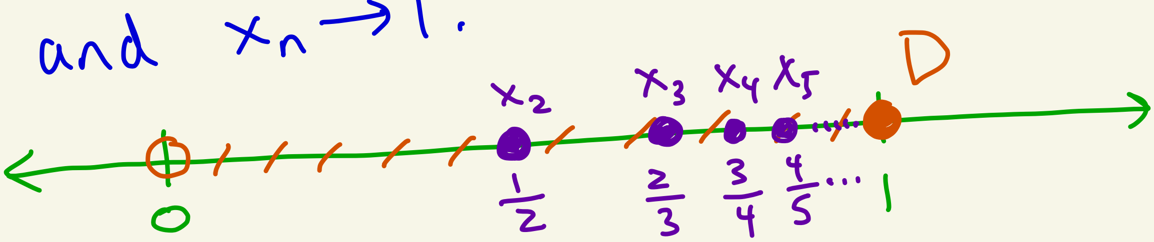


Thus, 2 is not a limit point of D.

Theorem: Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$.
Then: a is a limit point of D if and only if there exists a sequence (x_n) contained in D with $x_n \neq a$ for all n and $\lim x_n = a$. $\boxed{/}$

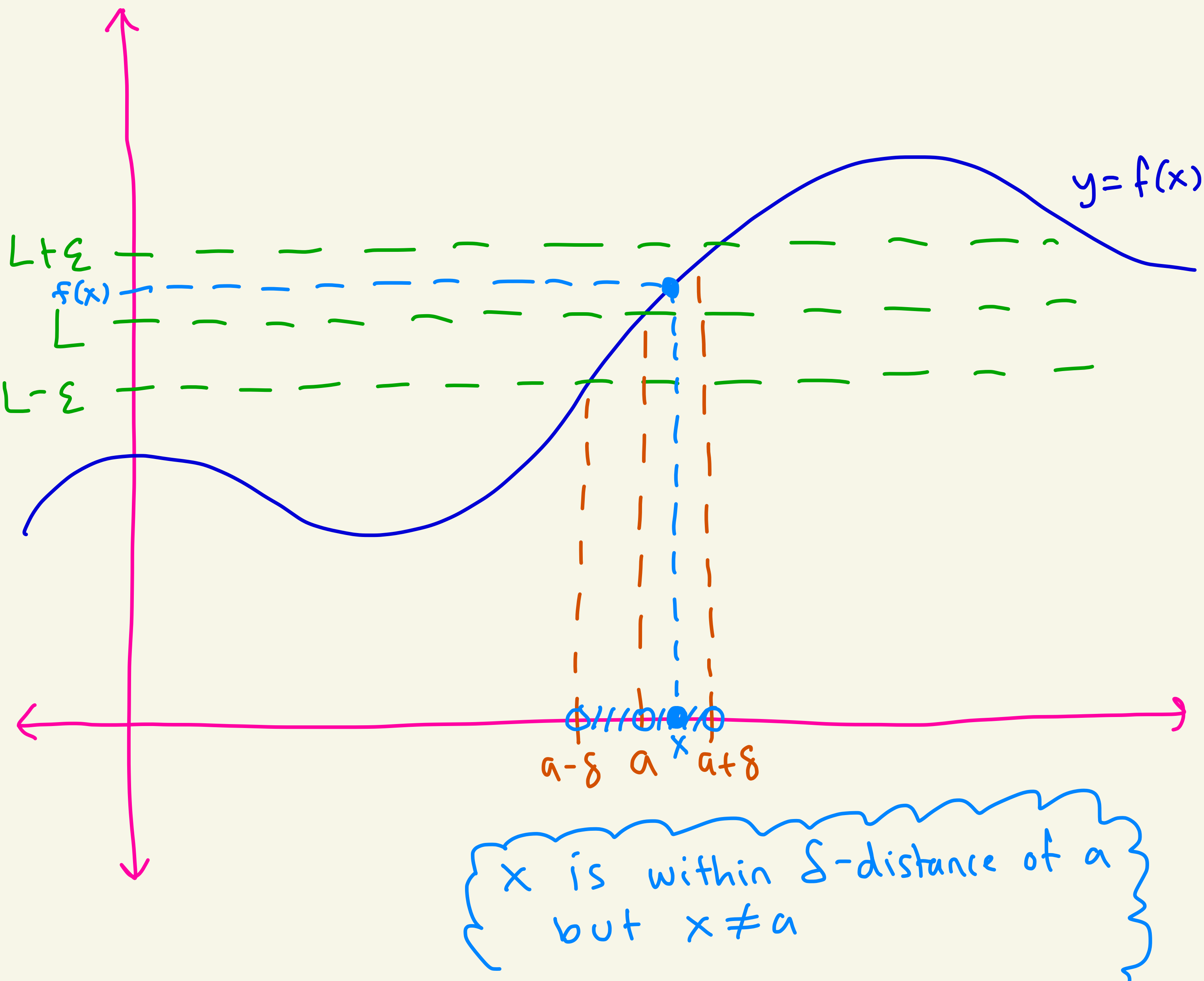
pf: HW

Ex: 1 is a limit point of $D = (0,1]$ because if we set $x_n = 1 - \frac{1}{n}$ with $n \geq 2$ then $(x_n)_{n=2}^{\infty}$ is contained in D, $x_n \neq 1$ for all n and $x_n \to 1$.



Def: Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$.
Let a be a limit point of D and $L \in \mathbb{R}$.
We write $\lim_{x \to a} f(x) = L$ if

for every $\varepsilon > 0$ there exists $\delta > 0$
so that if $x \in D$ and $0 < |x - a| < \delta$
then $|f(x) - L| < \varepsilon$.



Ex: Let's show that $\lim_{x \to 2} x^2 = 4$.

proof:

Let $\varepsilon > 0$.

We need to find $\delta > 0$ so that if $0 < |x-2| < \delta$, then $|x^2-4| < \varepsilon$

Note that $|x^2-4| = \underbrace{|x-2|}_{\text{we can control this with } \delta} \underbrace{|x+2|}_{\text{we need to bound this by restricting how big } \delta \text{ can be}}$

Suppose $\delta \leq 1$.

Then if $0 < |x-2| < \delta \leq 1$ we will get

$$
\begin{aligned}
|x+2| = |x-2+2+2| &= |x-2+4| \\
&\leq |x-2| + |4| \\
&\leq 1 + 4 \\
&= 5
\end{aligned}
$$

Thus, if $0 < |x-2| < \delta \leq 1$, then

$$|x^2-4| = |x+2||x-2| < 5|x-2|$$

Set $\delta = \min\left\{\frac{\varepsilon}{5}, 1\right\}$.

Then, if $0 < |x-2| < \delta$ we will get

$$|x^2-4| = |x+2|\,|x-2| < 5\delta \leq 5\,\frac{\varepsilon}{5} = \varepsilon.$$

$|x-2|<1$ $\qquad \delta \leq \frac{\varepsilon}{5}$

So if $0<|x-2|<\delta$, then $|x^2-4|<\varepsilon$. ∎

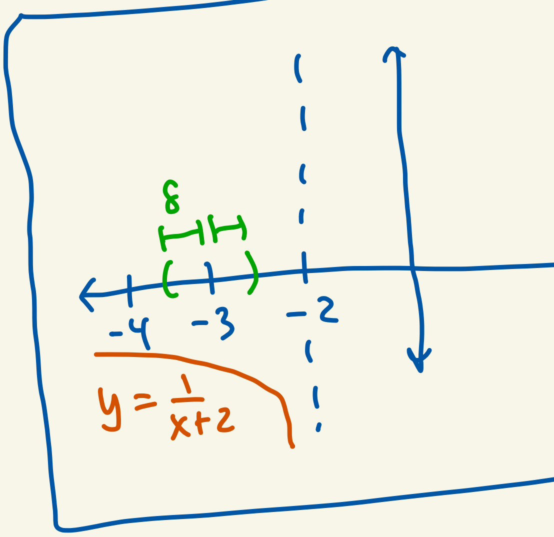
Ex: Let's show $\lim_{x \to -3} \frac{1}{x+2} = -1$

proof: Let $\varepsilon > 0$.

[We need to find $\delta > 0$ so that if $0 < |x-(-3)| < \delta$, then $\left|\frac{1}{x+2} - (-1)\right| < \varepsilon$]

Note that

$$\left|\frac{1}{x+2} - (-1)\right| = \left|\frac{1+x+2}{x+2}\right| = \left|\frac{x-(-3)}{x+2}\right| = \underbrace{|x-(-3)|}_{\text{this we can bound with } \delta} \cdot \underbrace{\left|\frac{1}{x+2}\right|}_{\text{make } \delta \text{ small enough to bound this}}$$



$\delta \le \frac{1}{2}$ ← we picked $\frac{1}{2}$ to avoid the asymptote at $x = -2$

Suppose $\delta \le \frac{1}{2}$.

Suppose $0 < |x-(-3)| < \delta \le \frac{1}{2}$

Then,

$$|x+3| < \frac{1}{2}$$

So,

$$-\frac{1}{2} < x+3 < \frac{1}{2}$$

giving

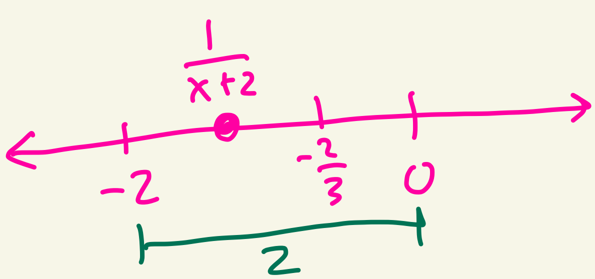
$$-\frac{3}{2} < x+2 < -\frac{1}{2}$$

which gives

$$-2 > \frac{1}{x+2} > -\frac{2}{3}$$

which gives

$$\left|\frac{1}{x+2}\right| < 2$$



Set $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$.

Then, if $0 < |x-(-3)| < \delta$ we get

$$\left|\frac{1}{x+2} - (-1)\right| = |x-(-3)| \cdot \left|\frac{1}{x+2}\right|$$
$$< \delta \cdot 2$$
$$\leq \frac{\varepsilon}{2} \cdot 2$$
$$= \varepsilon$$

So, if $0 < |x-(-3)| < \delta$, then $\left|\frac{1}{x+2} - (-1)\right| < \varepsilon$

$\blacksquare$

<u>Theorem:</u> Limits of functions are unique.

That is, if $f: D \to \mathbb{R}$ and a is a limit point of D with $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$

then $L_1 = L_2$.

<u>proof:</u> HW.

$\blacksquare$

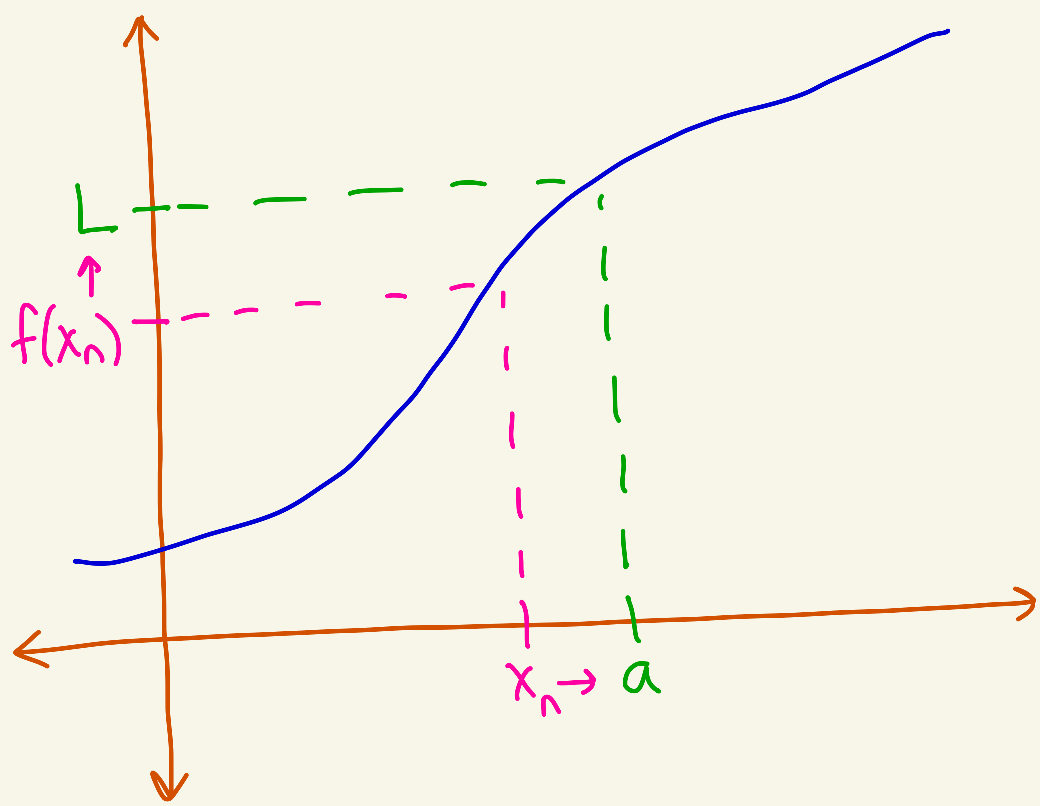
Function-sequence limit theorem:

Let $f: D \to \mathbb{R}$ and a be a limit point of D.

The following are equivalent:

① $\lim_{x \to a} f(x) = L$

② $\lim_{n \to \infty} f(x_n) = L$ for every sequence (x_n) contained in D with $x_n \neq a$ for all n and $x_n \to a$.



proof:

(① ⇒ ②) Suppose $\lim_{x \to a} f(x) = L$.

Let (x_n) be a sequence contained in D such that $x_n \neq a$ for all n and $x_n \to a$.

Let's show that $f(x_n) \to L$.

Let $\varepsilon > 0$.

Since $\lim\limits_{x \to a} f(x) = L$, there exists $\delta > 0$ so that if $0 < |x-a| < \delta$ and $x \in D$ then $|f(x) - L| < \varepsilon$.

Since $x_n \to a$ there exists N where if $n \geq N$ then $0 < |x_n - a| < \delta$.

since $x_n \neq a$ for all n

Thus, if $n \geq N$, then $|f(x_n) - L| < \varepsilon$.

We have shown that $\lim\limits_{n \to \infty} f(x_n) = L$

(② ⇒ ①) Suppose that $\lim\limits_{n \to \infty} f(x_n) = L$ for every sequence (x_n) contained in D with $x_n \neq a$ for all n and $x_n \to a$.

Let's show this implies that $\lim\limits_{x \to a} f(x) = L$.

Suppose to the contrary that $\lim\limits_{x \to a} f(x) \neq L$.

This implies that there exists a

particular $\varepsilon_0 > 0$ where no matter what $\delta > 0$ is chosen there exists $x \in D$ with $0 < |x-a| < \delta$ but $|f(x) - L| \geq \varepsilon_0$.

Consider $\delta_n = \frac{1}{n}$.

Then there exists a sequence (x_n) where for each n we have $x_n \in D$ and $0 < |x_n - a| < \frac{1}{n}$ but $|f(x_n) - L| \geq \varepsilon_0$

This gives us a sequence $x_n \to a$ with $x_n \neq a$ for all n but $\lim\limits_{n\to\infty} f(x_n) \neq L$.

Contradiction. $\blacksquare$

<u>Corollary</u>: Let $D \subseteq \mathbb{R}$ and a be a limit point of D. Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$. Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to b} g(x) = M$.

Then:

① $\lim_{x \to a} cf(x) = cL$

② $\lim_{x \to a} [f(x) + g(x)] = L + M$

③ $\lim_{x \to a} [f(x) g(x)] = LM$

④ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$ and $g(x) \neq 0$ for all $x \in D$.

<u>Proof:</u> We use the function-sequence limit theorem.

Let (x_n) be a sequence in D where $x_n \neq a$ for all n and $x_n \to a$.

By the function-sequence limit theorem

$\lim_{n \to \infty} f(x_n) = L$ and $\lim_{n \to \infty} g(x_n) = M$.

Thus, by the algebra of sequences theorem we get that

$$\lim_{n\to\infty} cf(x_n) = c\lim_{n\to\infty} f(x_n) = cL$$

$$\lim_{n\to\infty} [f(x_n)+g(x_n)] = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = L+M$$

$$\lim_{n\to\infty} [f(x_n)g(x_n)] = \left[\lim_{n\to\infty} f(x_n)\right]\left[\lim_{n\to\infty} g(x_n)\right] = LM$$

Since (x_n) was arbitrary, by the function-sequence limit theorem, we have proven ①, ②, ③.

For ④, suppose $M \neq 0$ and $g(x) \neq 0$ for all $x \in D$.
Since (x_n) is contained in D we have $g(x_n) \neq 0$ for all n.
By the algebra of sequences theorem we get

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \frac{L}{M}$$

By the function-sequence limit theorem this proves ④

$\blacksquare$
